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Chaotic Advection of Fluid Particles

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Chaotic advection of fluid particles

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Particle motion in a fluid can be chaotic even when the flow field is very simple from an eulerian point of view. This basic feature of fluid kinematics, known as chaotic advection, is reviewed and a number of applications are cited. The notion of a chaotic 'kinematic template' underlying dynamical processes is introduced and discussed. Some emerging directions of investigation for this application of chaos to fluid mechanics are indicated.

1. Introduction

In a well-known paper, published in the first volume of *J. Fluid Mech.*, Lighthill (1956) stated: 'Hydrodynamics has achieved impressive results by the simplification of concentrating on the steady (or nearly steady) field of flow velocities relative to a moving body, as specified in the Eulerian manner. This does not mean, however, that additional information, regarding the history of individual particles of fluid, is of no value.' Indeed, in many applications, such as fluid stirring, materials processing, pollutant dispersal and drug delivery, it is precisely the information inherent in the 'history of individual particles' that is of central interest.

The dichotomy to which Lighthill was alluding concerns what has come to be known as the eulerian and lagrangian representations of fluid flow. Truesdell (1954) reminds us that these attributions (first made by Dirichlet in 1860) to Euler and Lagrange are not historically accurate. Be that as it may, the nomenclature has stuck and concerns the following familiar issue. We may, on one hand, choose to describe a fluid in the way finite particle systems in classical mechanics are described by giving the coordinates of each fluid particle as a function of time. Let these functions be $\boldsymbol{X}(t;\boldsymbol{a})$, where $\boldsymbol{a}$ is a label that has to be continuously variable in the case of a fluid, and that may conveniently be taken as the initial position of the fluid particle in question, i.e. $\boldsymbol{X}(0;\boldsymbol{a}) = \boldsymbol{a}$. We may, on the other hand, choose to describe fluid flow in the way other fields, e.g. those arising in electromagnetism, are described. Then, if $\boldsymbol{x}$ denotes the continuously variable position vector within the fluid region and t denotes time, a vector function, $\boldsymbol{V}(\boldsymbol{x},t)$, dependent on space and time and known as the fluid velocity field is introduced. In fluid mechanics the particle description is referred to as the lagrangian representation, while the field description is known as the eulerian representation.

There is an important connection between the two representations. At time t the particle passing through $\boldsymbol{x} = \boldsymbol{X}(t;\boldsymbol{a})$ has velocity

$$\partial \boldsymbol{X}(t;\boldsymbol{a})/\partial t = \boldsymbol{V}(\boldsymbol{X}(t;\boldsymbol{a}),t). \tag{1}$$

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This equation essentially expresses the definition of the velocity field in terms of sampled velocities of moving fluid particles.

Another way of obtaining the same result is to express the 'rate of change following a particle', an essentially lagrangian quantity, in terms of eulerian fields and their derivatives. Thus, if p is a quantity whose distribution in space and time is described by the field $p(\boldsymbol{x}, t)$, then the rate of change of p following a particle is given by

$$\mathrm{D}p/\mathrm{D}t = \partial p/\partial t + \boldsymbol{V}\cdot\nabla p. \tag{2}$$

This quantity, designated by capital D when writing the derivative, is known as the material derivative. The vector generalization of (2) is

$$\mathrm{D}\boldsymbol{P}/\mathrm{D}t = \partial \boldsymbol{P}/\partial t + \boldsymbol{V}\cdot\nabla \boldsymbol{P}. \tag{3}$$

In particular, if $\boldsymbol{P}$ is the position vector field, $\boldsymbol{P} = \boldsymbol{x}$, the definition (3) yields the 'eulerian version' of (1), namely

$$\mathrm{D}\boldsymbol{x}/\mathrm{D}t = \boldsymbol{V}(\boldsymbol{x}, t). \tag{4}$$

(The interpretation of this equation as equivalent to (1) then relies on understanding the proper physical meaning of the material derivative.)

Either way the velocity field induces a 'natural motion' of fluid particles according to the system of ordinary differential equations (1) or (4). (We use the term 'natural motion' in the sense of Khinchin (1949) in his discussion of the phase space motion of a hamiltonian system.) To press this point let θ be a scalar field that is tracked particle by particle, so that it may be written

$$\theta(\boldsymbol{x}, t) = \int \theta(\boldsymbol{a})\, \delta(\boldsymbol{x} - \boldsymbol{X}(t;\boldsymbol{a}))\, \mathrm{d}\boldsymbol{a}, \tag{5}$$

where $\theta(\boldsymbol{a})$ is the value of θ for the particle which at $t = 0$ is at $\boldsymbol{X} = \boldsymbol{a}$ and δ denotes the Dirac δ-function. Then (1) is equivalent to the field equation

$$\partial\theta/\partial t + \boldsymbol{V}\cdot\nabla\theta = 0. \tag{6}$$

Although this example suggests that there is a simple correspondence between the equations of motion for a fluid written in the eulerian and the lagrangian representations, and the choice is largely a matter of taste, analytical tradition has clearly favoured the eulerian representation, and considered the lagrangian version to be analytically intractable. There is much to support this point of view. Nevertheless, with the recent surge of interest in computational fluid dynamics the lagrangian representation has gained in significance and applicability, since such formulations are often extremely elegant and intuitive, and the analytical complexity is obviated by the introduction of appropriate algorithms. I cannot give a full discussion of this topic here. Some representative references are Buneman (1982), Hockney & Eastwood (1988), and Leonard (1980, 1985).

In D dimensions there are D coupled ordinary differential equations (ODEs) in the system (1) or, equivalently, (4). The motion of particles in a given fluid mechanical system will only be described by these ODEs if the particles are sufficiently light, inert, passive, etc., that they 'move with the flow'. In many cases of interest this will not occur. The particles to be tracked may have inertia, or be subject to diffusion, or there may be some physical characteristic that differentiates particles of the advected substance from the advecting fluid.

I shall use the following terminology. Equations (1) or (4) with a prescribed velocity field $\boldsymbol{V}$ will be called the *advection equations*. A particle moving according to these equations is said to be subject to passive advection. In cases where the advection equations only provide a background for dynamically interesting action, but do not fully describe the motion of the particles of interest, I shall refer to the natural motion given by (1) or (4) as the *kinematic template* of the flow. In such cases the advection equations provide a base state of motion from which the motion of interest may deviate to a greater or lesser degree.

For example, considering the case of flow about a uniformly translating object, such as a vehicle, the streamline pattern about the vehicle (in its rest frame) provides the kinematic template. The actual trajectory of a massive particle (e.g. a stone or an insect that hits the vehicle) deviates from the particle paths of the kinematic template due to inertia. Similar issues arise in problems of dust and particle motion around aircraft engines and in the performance of particle separators. The degree to which the kinematic template is followed by physical particles is also a recurring problem for flow visualization studies. The article by M. R. Maxey (this issue) shows that pursuing this subject by considering modifications of (1) opens up a host of interesting problems related to and extending the ideas presented herein.

2. Regular and chaotic advection

The advection equations are a set of coupled ordinary differential equations. The cases $D = 2$ and 3 are the most interesting from the point of view of applications. The equations have the following formats:

$$D=2: \quad \mathrm{d}x/\mathrm{d}t = u(x,y,t), \quad \mathrm{d}y/\mathrm{d}t = v(x,y,t), \tag{7}$$

$$D=3: \quad \mathrm{d}x/\mathrm{d}t = u(x,y,z,t), \quad \mathrm{d}y/\mathrm{d}t = v(x,y,z,t), \quad \mathrm{d}z/\mathrm{d}t = w(x,y,z,t). \tag{8}$$

From these we immediately have the following conclusions.

1. The advection equations are integrable for steady flow in two dimensions.
2. The advection equations may be non-integrable, with chaotic particle trajectories, for unsteady flow in two dimensions and for steady or unsteady flow in three dimensions.

Integrable cases of the advection equations lead to what I shall call *regular advection*. Non-integrable cases imply motion that I call *chaotic advection*. Some writers have used the names 'lagrangian turbulence' (Dombre *et al.* 1986; Chaiken *et al.* 1986, 1987; McLaughlin 1988), 'chaotic mixing' (Chien *et al.* 1986), and 'chaotic convection' (Ott & Antonsen 1988) for this régime. Regarding the first of these it is, as Drazin (1988) (see also Cox *et al.* 1990) has so eloquently put it, 'felt improper to label "turbulent" processes that manifestly are not.' Regarding the second, I prefer, following Eckart (1948), to reserve the term mixing for processes that involve the molecular diffusivity or other inter-material transport processes. The third name causes confusion with common usage in the thermal convection literature, where 'chaotic convection' means a flow that is chaotic in the eulerian sense. It should be stressed that chaotic advection is a result of flow kinematics. The momentum equation of the flow does not enter in the above considerations. In particular, chaotic advection may occur in flows at low Reynolds number, including Stokes flows, and such flows provide important illustrations and areas of application of the subject (see several references given in table 1).

Table 1. *Representative papers on chaotic advection listed by application area, flow dimensionality, and remarks on the nature of the investigation*

area of application	2-D	3-D	other remarks: A = analytical, E = experimental, N = numerical	reference
advection–diffusion	*	—	separation device (A, N)	Aref & Jones (1989)
	—	*	anomalous dispersion (A, N)	Jones & Young (1989)
cascade dynamics/GFD	*	—	(A, N)	Falcioni *et al.* (1988)
	*	—	(A, N)	Pierrehumbert (1988)
'clean room'	—	*	(N, A)	Lichter *et al.* (1990)
coagulating particles	*	—	(N, A)	Muzzio & Ottino (1988)
convection	—	*	(A, N)	Arter (1983)
	*	—	(E, N)	Solomon & Gollub (1988)
	—	*	(A, N)	Holm *et al.* (1990)
	*	—	Earth's mantle (A, N)	Kellogg & Turcotte (1990)
effects of particle size	*	—	(A, N)	McLaughlin (1988)
efficiency	*	—	(A, N)	Khakhar & Ottino (1986)
floats, buoys	*	—	(E, A)	Osborne *et al.* (1986)
general	*	—	'blinking vortex' (A, N)	Aref (1984)
	*	—	model flows (A, N)	Khakhar *et al.* (1986)
	*	*	(A, N)	Aref *et al.* (1988)
	*	*	(E, A, N)	Ottino (1989*a*, *b*, 1990)
	*	*	(A, N)	Cox *et al.* (1990)
incompressibility	*	—	(A, N)	Hénon (1969)
inviscid flow	—	*	'ABC flow' (A, N)	Hénon (1966)
	—	*	'ABC flow' (A, N)	Dombre *et al.* (1986)
	*	—	source/sink (A, N)	Jones & Aref (1988)
	*	—	'ABC map' (A, N)	Feingold *et al.* (1988)
morphology	*	—	whorls and tendrils (A)	Berry *et al.* (1979)
	*	—	spirals (A)	Moffatt (1984)
	*	—	spirals (A, N)	Gilbert (1988)
	*	—	(E, A)	Ottino *et al.* (1988)
	*	*	role of symmetry (A, N)	Beloshapkin *et al.* (1989)
pipe flow	—	*	twisted pipe (A, N)	Jones *et al.* (1989)
sedimentation	*	—	(A, N)	Smith & Spiegel (1985)
Stokes flow	*	—	'eccentric cylinders' (A, N)	Aref & Balachandar (1986)
	*	—	'eccentric cylinders' (E, A, N)	Chaiken *et al.* (1986, 1987)
	*	—	'cavity/extruder' (E, A)	Chien *et al.* (1986)
	—	*	'partitioned pipe mixer' (A, N)	Khakhar *et al.* (1987)
	—	*	spherical 'droplet' (A, N)	Bajer & Moffatt (1990)
	*	—	'eccentric cylinders' (E, A, N)	Swanson & Ottino (1990)
tidal flows	*	—	Wadden Sea (A, N)	Zimmerman (1986)
	*	—	(A, N)	Pasmanter (1988)
vortex motion	*	—	restricted four-vortex (A, N)	Aref & Pomphrey (1980)
	*	—	restricted four-vortex (A)	Ziglin (1980)
	*	—	(A, N)	Aref *et al.* (1989)
wave motion	*	—	(A, N)	Knobloch & Weiss (1987)
				Ramshankar *et al.* (1990)

The notion of chaotic advection was introduced some 25 years ago by Arnol'd (1965) and Hénon (1966) in the context of a class of steady (Beltrami flow) solutions to the three-dimensional Euler equations in a periodic cube, since called the 'ABC flows'. Their results apparently generated little interest in the fluid mechanics literature of the time, possibly because they were seen as pertaining to certain

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special, essentially unrealizable instances of inviscid flow, and have only been revisited and extended rather recently (Dombre *et al.* 1986) (see also Arter 1983). On the other hand, a considerable amount of activity has taken place in the closely related problem of charged particle motion in electric and magnetic fields of interest to plasma physics, astrophysics and to the performance of particle accelerators for high energy physics (see the discussion in Mackay & Meiss 1987).

The result that two-dimensional steady flow leads to regular advection is valid for both incompressible and compressible flow fields. The incompressible case will be discussed below since it admits further elaboration. The general result follows from the so-called Poincaré–Bendixon theorem (cf. Stoker 1950). It is, of course, the integrability of (1), (4) or (8) that is the key issue, not the dimensionality. Three-dimensional advection may be integrable because of the possibility of a reduction to similarity variables, or because of some symmetry in the problem. There are several examples in the literature of such special cases (cf. Cantwell 1981 *a*, *b*).

Much of the recent interest in the application of chaotic particle motion to the stirring of fluids may be traced to work by the author (Aref 1982 *a*, *b*, 1984) in which the ideas are explained using a simple example from unsteady two-dimensional flow. Apart from the intrinsic scientific interest this work showed that chaotic advection is immediately accessible in easily constructed flow fields and, thus, suggested the general applicability of this mechanism to a variety of situations. Indeed, in recent years we have seen a surge of interest in producing, identifying and elucidating chaotic advection in a wide variety of flow situations, including stirring and mixing devices in chemical engineering processes, dispersal in atmospheric and oceanographic flows (e.g. in tidal channels), mixing and transport phenomena in biological systems (e.g. pulsatile flow in corrugated channels and flow in twisted tubes), and mixing in convective flows both in nature and technology. New applications appear regularly. A list of representative topics and references is summarized in table 1. The level of interest is further evidenced by the appearance of a monograph (Ottino 1989 *b*) and by increasing representation of work on chaotic advection in the proceedings of several recent conferences. (See Guyon *et al.* (1988), Moffatt & Tsinober (1990) and the forthcoming special volume of *Phys. Fluids* A containing the proceedings of the IUTAM Symposium 'Fluid mechanics of stirring and mixing'.)

Some important related works, that discuss the changes in topology that come about by bifurcations in the streamline patterns of steady flows but do not consider the possibility of chaos, are Perry & Fairlie (1974) (see also Perry & Chong 1987) and Cantwell (1981 *a*, *b*). That pathlines and streaklines in simple two-dimensional unsteady flows can become very complicated is, of course, well known. By way of example consider the paper by Hama (1962), followed many years later by Williams & Hama (1980), on the counter-intuitive nature of pathlines and streaklines in unsteady flows. See also the paper by Regier & Stommel (1979). Using different tools and concepts the papers by Onsager (1949) and Welander (1955) pursue the analogy of two-dimensional incompressible flow of a fluid and the allegedly ergodic phase space flow introduced in gibbsian statistical mechanics. Below I shall return in more detail to this analogy, which motivates the use of the term 'natural motion'.

It is at once encouraging and depressing that even laminar flow kinematics can demand the full machinery of chaotic behaviour. It is encouraging since many workers in fluid mechanics know that considerable complexity can ensue in the distribution of an advected scalar even if the variation of eulerian quantities is 'simple'. The identification of these phenomena with chaos in the technical sense of dynamical systems theory has provided a new framework for treating such cases, and a fresh complement to more traditional descriptions based, among other things, on the general theory of continuum mechanics (see Ottino (1989*b*) for discussion of this material). In this way it has propelled aspects of low Reynolds number fluid stirring from their status of applied and empirical subjects to the front lines of scientific inquiry. Experiments on chaotic advection using fluorescent dyes produce striking images that have generated considerable excitement because the complexity of the system is immediately visible in real space, rather than being associated with an abstract structure in phase space (see the explicit, formal statement for two-dimensional flow below). Indeed, illustrations of chaotic advection appear on the covers of *Nature* and *Scientific American* in conjunction with the articles by Ottino *et al.* (1988), Beloshapkin *et al.* (1989) and Ottino (1989*a*). It is, on the other hand, depressing for someone interested in turbulent flow to realize that even the kinematics of laminar advection can in the words of Arnol'd be 'beyond the capability of modern science' (cf. Arnol'd 1978, p. 22).

In the case of two-dimensional incompressible flow an interesting connection to the theory of hamiltonian dynamics arises. Let the condition that the flow is incompressible be accounted for by the standard introduction of a streamfunction $\psi(x, y, t)$ such that

$$u = -\partial\psi/\partial y, \quad v = \partial\psi/\partial x. \tag{9}$$

Equations (7) now become

$$\mathrm{d}x/\mathrm{d}t = -\partial\psi/\partial y, \quad \mathrm{d}y/\mathrm{d}t = \partial\psi/\partial x. \tag{10}$$

These have the form of Hamilton's canonical equations. The streamfunction plays the role of the hamiltonian, and the conjugate variables are the particle coordinates x and y. For this system 'phase space' is 'configuration space'. This identification of the kinematics of two-dimensional fluid flow with the dynamics of a one-degree-of-freedom hamiltonian system opens the way for wholesale application of tools from dynamical systems theory, including the role of periodic points, 'horseshoes', homoclinic and heteroclinic tangles, etc. Much of this material is already well developed in the monograph and textbook literature (Arnol'd & Avez 1968; Moser 1973; Lichtenberg & Lieberman 1983; Guckenheimer & Holmes 1983; MacKay & Meiss 1987).

The correspondence of the phase space motion of a hamiltonian system with particle motion in incompressible fluid flow is frequently noted in discussions of Liouville's theorem in statistical mechanics (cf. Landau & Lifshitz 1980) and goes back to the origins of that subject. The notion of chaotic advection, in a sense, uses this analogy 'in reverse' comparing the more intuitive real space motion of the fluid with the abstract phase space motion of a dynamical system, and then appealing to the understanding of phase space chaos in hamiltonian systems that has been evolving steadily since the seminal work of Poincaré. There is an apparently important point here, which, however, has not so far played much of a role in fundamental studies or applications of chaotic advection. This is the observation that while the 'right-hand sides' of a general dynamical system may be prescribed essentially arbitrarily, the advecting flow responsible for chaotic advection, i.e. for two-dimensional flow the hamiltonian/streamfunction, is constrained to satisfy an equation of motion for the fluid, such as the Navier–Stokes equation. One may expect this constraint to place certain limits on the chaotic nature of particle trajectories, but we are not aware of any substantial results in this direction. (In the most popular application, two-dimensional unsteady Stokes flow, the unsteadiness arises from the motions of boundaries, and these may again be specified arbitrarily. Chaotic advection by three-dimensional steady flow is more constrained in the sense being discussed here.)

The appearance of chaotic particle trajectories in the advection problem explains why fractal structure can appear in the signature left by a passive marker. For an exposition of the mathematics and physical science applications of fractals see the monograph by Feder (1988). With particular reference to fluid mechanics we note the brief review article by Turcotte (1988). We include here both observations of fractality in clouds (Lovejoy 1982) and tracers in shear flows (Sreenivasan & Meneveau 1986), and observations of individual trajectories (Osborne *et al*. 1986). The utility of fractal dimension studies of such signals is still a matter of debate. While the presence of fractals (in one form or another) is presumably beyond question, the ability of fractal dimension concepts to distinguish flows, indeed to distinguish laminar from turbulent agitation, is still unsettled as, indeed, the phenomenon of chaotic advection suggests it must be. In this context the method of wavelets (Holschneider 1988; Arneodo *et al*. 1988; Argoul *et al*. 1989) seems promising. For example, in Aref *et al*. (1989) the computed agitation of a region of fluid by a few hundred point vortices is compared to experimental images of the cross-section of a round jet due to D. Liepmann and M. Gharib. The similarity of the fractal nature of the contours in the two sets of images is considerable, and the dilemma from the point of view of the utility of fractals is that the underlying dynamics differs profoundly. (In this context see also the recent study by Vassilicos (1990).)

3. Numerical considerations

The elucidation of chaotic advection has been strongly aided by the ability to perform precision numerical experiments on the finite-dimensional dynamical system presented by the advection equations. A number of considerations, common to much of computational fluid dynamics, are highlighted by this particular application. The main common thread in all of these is the issue of how to decide between chaotic behaviour arising from numerical sources, and chaos that is inherent to the equations being integrated. Closely related to this problem is that of assessing how far into the chaotic régime one can follow the advection of a passive scalar computationally as compared with physical experiments.

The initial investigations, both of advection by ABC flow (Hénon 1966; Dombre *et al*. 1986), three-dimensional convection (Arter 1983), and two-dimensional unsteady flows (Aref 1984; Aref & Balachander 1986; Chaiken *et al*. 1986, 1987) all used flow fields that were given explicitly as functions of the coordinates everywhere in space. Hence, in (1) or (4) there is no need to interpolate the velocity field in space when solving the advection problem numerically, and the only source of computational error is the truncation error in the time integration. This error is generally well controlled, at least for runs that are not too long, and can, in any event, be assessed by time reversal of the flow, or by changing the tolerance in an adaptive step size algorithm. Recently, the interesting technique of using symplectic integrators (Channell & Scovel 1990) has been adapted to the convection problem (Holm *et al*. 1990), further reducing concerns over this source of error. We may also mention that for some of the simple, pulsed two-dimensional flows a reduction to an explicitly given mapping is possible, and the 'time step' in question then increases to an entire half-cycle of the pulsed flow. All this does not, however, remove the generic and deep concern about the ability of numerical experiments to follow chaotic trajectories. The literature (McCauley & Palmore 1986; Hammel *et al.* 1987) is still several steps removed from addressing as complex a problem as chaotic advection, let alone the true nature of the 'numerical turbulence' observed in large-scale simulations of fluid flow at Reynolds numbers of a few thousands.

I am convinced that the chaotic behaviour observed in numerical simulations of chaotic advection is, indeed, a genuine physical effect. This conviction is amplified by our understanding of the general framework of chaos in dynamical systems with two degrees of freedom, into which these problems seem to fit, and, of course, by the verification through laboratory experiments. Nevertheless, other issues arise when chaotic advection is viewed from the perspective of applications. Foremost among these is the obvious one that not all flows of practical interest are going to be known in closed analytical form. For a case such as flow in a twisted pipe (Jones *et al.* 1989), for example, the solutions used are perturbation series, and one faces the problem of how many terms to retain in the series. This problem is analogous to that of how fine a grid to use if the flow is to be obtained from a conventional finite difference or finite element solution to the governing equations of motion.

For Stokes flow a particularly suitable and elegant approach is available in the so-called boundary integral method. In two dimensions recent progress with this kind of method has led to convincing and efficient calculations of Stokes flow across ridges and cavities (Higdon 1985) and through corrugated channels (Pozrikidis 1987*a*) including channels with deformable walls (Pozrikidis 1987*b*). These results suggest that combination of the boundary integral method and the advection equations provides a versatile framework for precision calculations of advection by Stokes flows (H. Aref & C. Pozrikidis, unpublished work). The approach outlined was motivated by consideration of how to perform numerical simulations of the experiments of Sobey (1985). The boundary integral method shares with the explicitly given flow fields the desirable property of being given analytically at every point within the flow without need for interpolation. The expression obtained is, of course, an approximation with an accuracy governed by the density of source points on the flow boundary and the quadrature rule used for evaluating the flow at the field point. However, this kind of approximation generally gives a series of smooth approximants, which can, moreover, be constructed to produce the flow field with spectral accuracy. Hence, in this type of application the need to interpolate an eulerian field known at discrete gridpoints of a spatial mesh is avoided.

The close correspondence that exists between the stirring by laminar flow of a passive scalar and the phase space dynamics of the dynamical system that has the flow streamfunction as its hamiltonian, and the almost arbitrarily fine spatial resolution obtainable with certain combinations of marking dye and advecting fluid, suggests that the phenomenon of chaotic advection may yield 'analogue computers' for nonlinear dynamics of considerable interest. Experiments (see, for example, Chaiken *et al.* 1986) have shown that remarkable agreement can be achieved between flow visualization data and computational results in ranges where both are applicable. However, as in other chaotic systems, the repeated folding and thinning of fluid layers that is characteristic of chaotic advection quickly taxes numerical methods for following the ensuing structure (Franjione & Ottino 1987).

We have indicated that chaotic advection plays a significant role in its own right and has direct applications, and in this section we have briefly touched upon issues

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concerning the elucidation of chaotic advection effects via numerical experiments. In the next two sections we pursue the notion of the 'kinematic template' mentioned previously, i.e. discuss cases in which the 'natural motion' is simply a background against which dynamical processes occur. This means that we must consider problems in which the particle paths of interest deviate from streamlines. One of the most important instances of this occurs in the evolution of vorticity distributions in essentially inviscid flow. In this context chaotic advection may have something to offer in the discussion of turbulent flows dominated by 'coherent structures', which are typically well endowed with vorticity.

4. Regular and chaotic point vortex motion

The equation of motion for the vorticity field, sometimes called Helmholtz' equation, is one of the clearest examples of the significance and power of the lagrangian viewpoint in fluid mechanics. The equation concerns the vorticity field $\boldsymbol{\omega} = \nabla \times \boldsymbol{V}$ and states that

$$\mathrm{D}\boldsymbol{\omega}/\mathrm{D}t = (\boldsymbol{\omega}\cdot\nabla)\,\boldsymbol{V}+\nu\nabla^2\boldsymbol{\omega} \tag{11}$$

in viscous, incompressible fluid. The interpretation of the right-hand side in terms of reorientation, stretching and diffusion of vortex elements is well known (cf. Batchelor 1967).

Although there are problems of interpretation in three dimensions (Saffman & Meiron 1986; Winckelmans & Leonard 1988), we may formally consider solutions to (9) that are sums of δ-function singularities of the $\boldsymbol{\omega}$-field. The equations of motion for these bear a similar relation to (11) (with $\nu = 0$) as (1) does to the field equation (6).

The case of parallel vortex filaments, or equivalently 'point' vortices in the plane, is classical (Helmholtz 1858) and has an immediate relation to the problems discussed in §2. If the point vortices are located at $z_\alpha = x_\alpha + \mathrm{i}y_\alpha$ and have circulations Γ_α, $\alpha = 1, \dots, N$, their equations of motion in ideal fluid, otherwise at rest, are

$$\frac{\mathrm{d}\bar{z}_\alpha}{\mathrm{d}t} = \frac{1}{2\pi\mathrm{i}}\sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{z_\alpha - z_\beta}, \tag{12}$$

where the overbar denotes complex conjugation, and the prime on the summation sign signifies omission of the singular term $\beta = \alpha$.

In this case (5) and (6) are satisfied by the sole non-vanishing component of vorticity, ζ. However, ζ is not independent of the velocity field (as was the passive scalar field θ of (5) and (6)). One must take account of the defining relation

$$\zeta = \partial v/\partial x - \partial u/\partial y = \nabla^2\psi, \tag{13}$$

the last equality arising from introduction of the streamfunction as in (9).

The equation of motion for a passively advected particle at $z(t)$ moving in the flow field set up by N vortices is

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}t} = \frac{1}{2\pi\mathrm{i}}\sum_{\beta=1}^{N} \frac{\Gamma_\beta}{z - z_\beta} \tag{14}$$

where (12) determines the evolution of the $z_\beta(t)$.

Analytical and computational work points to the following main result.

The passive advection problem with N vortices is integrable, if, and only if, the vortex interaction problem with N+1 vortices is integrable.

This result is independent of the actual values of the circulations. More significantly, it seems to carry over to cases of point vortex motion in bounded domains. The primary references are Gröbli (1877), Synge (1949), Novikov (1975), Aref (1979), Novikov & Sedov (1978, 1979), Aref & Pomphrey (1980, 1982), Eckhardt (1988*a*) and Eckhardt & Aref (1988). For reviews see Novikov (1980), Aref (1983) and Aref *et al.* (1988*a*). For treatment of special integrable cases see Eckhardt (1988*b*), Rott (1989, 1990) and Aref (1989).

The implication that if the $(N+1)$-vortex problem is integrable, then the motion of a passive particle in the field of N vortices is also integrable is, of course, trivial since a passive particle appears as the limiting case of a vortex with vanishing circulation. The other implication is more interesting and seems, in general, to be difficult to prove directly from the equations of motion. I am not aware of an analytical proof of this result. The motion of a passive particle in the field of three interacting point vortices – an integrable problem in the infinite plane – provided an important, early example of chaotic advection by unsteady, two-dimensional flow (Aref & Pomphrey 1980; Ziglin 1980).

It is clear that (14) 'inherits' the hamiltonian structure given by (10). It is interesting to note that (12) also possesses a hamiltonian structure, a fact noticed by Kirchhoff more than a century ago, and extended to motions of point vortices in bounded domains through work by Routh, Lagally and others, finally integrated into a general framework in the monograph of Lin (1943), which gives a lucid statement of the entire theory and its historical development.

5. Outlook

Although the appearance of chaotic trajectories in the motion of fluid particles in simple laminar flows is a pleasing and useful result both for fluid mechanical analysis and for the applicability of chaos in systems with a few degrees of freedom, the question of how important such a correspondence is for general flows may legitimately be asked. The fact that Stokes flow experiments and even practical devices can be constructed to provide laboratory realizations of chaotic systems is significant, but pertains only to a rather limited part of the spectrum of fluid flows. One may worry that, in general, chaotic trajectories are insignificant, and that the subject of chaotic advection is mainly of academic interest.

The situation to be addressed is a flow in which the 'natural motion' displays chaotic advection, yet the particles of interest deviate from streamlines due to various forces, effects and phenomena that obviate the description in terms of a passive scalar. In this situation the chaotic advection acts as a complex backdrop against which the motion of interest takes place. This is the notion of the kinematic template, and the general challenge is to determine when the complexity of this template gives relevant insight into the particle motion at issue. The case of point vortex motion was singled out above because it appears to be an instance where there is a very direct and intimate connection between the kinematics and the full dynamical problem. In general, we cannot expect to be so fortunate.

It is still too early to state definite conclusions regarding the problem area outlined. Only a few examples have been investigated in any detail, and much work remains to be done before a coherent picture will emerge. The most immediate examples are those in which additional deterministic terms involving the velocity field and/or some intrinsic property of the advected particles appear on the right-hand sides of (1) or (4). For example, one may add a sedimentation velocity along a line fixed in space (Smith & Spiegel 1985) or a more complete dynamical specification involving the changes in a vector or tensor describing the particle geometry (McLaughlin 1988; M. R. Maxey, this issue). In these cases it is clear that the role of the velocity field in determining regularity or chaos of passive advection is played by the field that appears on the right-hand sides of (1) or (4). It is, in general, difficult to predict whether a given modification will yield chaotic or regular motion. On the basis of general notions of structural stability and generic behaviour, one would expect it to be relatively easy to produce modifications that would render a regular advection problem chaotic. It would be a rare occurrence to see a chaotic advection problem become regular by the addition of further interaction terms. However, such global statements are not of great interest. The key problem is whether for a given application the picture obtained from passive advection studies yields any understanding of the 'full' problem. Chaotic advection studies can help identify cases where complicated behaviour associated with a chaotic 'kinematic template' have been erroneously attributed to interactions.

Examples that most readily come to mind in this context are from two-dimensional vortex dynamics. The motion of concentrated vortices is expected to resemble that of a corresponding set of point vortices, and is thus chaotic if there are sufficiently many vortices (Aref 1983). Hence, attributing such motions to core deformations or profiles would be misleading. More interesting is recent work on the dynamics of finite area vortices, where the spatial distribution of weak vorticity streamers during merging and evolution in a strain field appear to have an interpretation in terms of chaotic advection (Polvani & Wisdom 1990*a*, *b*). It is possible that also the filamentation from isolated, finite area vortices (Dritschel 1988) can be understood in terms of the chaotic advection of vorticity around the fixed point structure of the advection pattern set up at the rim of a rotating vortex patch. Such interpretations may extend to the subject of vortex reconnection in three dimensions.

Another, in some sense complementary, way of perturbing the passive scalar picture is to add a stochastic term to the deterministic advection field in (1) or (4). This allows diffusion to be accommodated in a lagrangian framework, and basically substitutes for the advection equations a set of Langevin equations. Primarily, diffusion plays a role for regular advection, since in the strongly chaotic régime the stochasticity due to a stochastic (brownian motion) model of diffusion simply abets the stochastic transport due to the deterministic velocity field. However, the general picture is that the advection equations divide the flow volume into regular and stochastic, namely the KAM tori and the chaotic sea, and the addition of diffusion allows particles to cross the boundaries between these two regions, a process that is strictly forbidden in the non-diffusive limit. For this reason diffusion can also have an important effect on chaotic advection. A couple of studies have attempted to elucidate this coupling (Jones *et al.* 1989; Jones & Young 1989). The work of Aref & Jones (1989) is concerned with the competition of a reversible, deterministic, Stokes flow component and an irreversible, brownian motion diffusive process.

A related application is to particle coagulation in an agitated fluid. It is well known that agitation of a coagulating mixture enhances the rate of precipitation, and one might speculate that chaotic advection would more frequently bring particles into close proximity than regular advection, and hence would be more effective in precipitating the coagulated material. The single available study of this application (Muzzio & Ottino 1988) makes a start on this problem, but much remains to be done.

The issue of spatial structures to be expected due to chaotic advection was addressed by Berry *et al.* (1979) and still holds considerable challenge, in particular in three dimensions. Moffatt (1984) suggested a connection between the signature due to an elliptic fixed point (basically a spiral) and the observed spectrum, and this has been elaborated by Gilbert (1988). These studies are, of course, closely related to the emergence of a level set with a fractal dimension mentioned previously. Further elucidation of the interplay between the configuration of periodic points, regions of chaos, spatial structures and exponents in spectra may pave the way for applications of the ideas presented here to turbulent flows.

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